

Numerically Computable A Posteriori-Bounds for SPDEs

Dirk Blömker*, Minoo Kamrani†

March 6, 2017

Abstract

The aim of this paper is the derivation of an a-posteriori error estimate for a numerical method based on an exponential scheme and spectral Galerkin methods. We obtain analytically a rigorous bound on the mean square error conditioned to the calculated data, which is numerically computable and relies on the given numerical data. Thus one can check a-posteriori the error for a given numerical computation without relying on an asymptotic result.

All estimates are only based on the numerical data and the structure of the equation, but they do not use any a-priori information of the solution, which makes the approach applicable to equations where global existence of solutions is not known. For simplicity of presentation, we develop the method here in a relatively simple situation of a stable one-dimensional Allen-Cahn equation with additive forcing.

1 Introduction

A-posteriori analysis of deterministic PDE (partial differential equations) is a well developed tool. See for example the book [18] or the results for Allen-Cahn and related equations [3, 2, 10]. The strength of the method is usually the derivation of error indicators for the refinement of meshes in adaptive schemes. See [17] for an example in a stochastic setting.

Also for SPDEs (stochastic PDEs) there are recent results on a-posteriori analysis. The results of [8, 13] use a-posteriori estimates in polynomial or Wiener-chaos expansion, and the results of [19, 20] show a-posteriori mean square error estimates, where the whole numerical approximation is known (at least several moments of it).

In our work we follow a different more path-wise approach. We measure the error in mean square conditioned on the calculated numerical data.

*Institut für Mathematik Universität Augsburg, 86135 Augsburg, Germany, E-MAIL: dirk.bloemker@math.uni-augsburg.de

†Department of Mathematics, Faculty of Sciences, Razi University, Kermanshah, Iran

Given a single realization of the numerical approximation, without using a-priori information on the solution we show analytic bounds, that can be calculated numerically, and guarantee a-posteriori that the true solution is close to the one realization of the numerical approximation, which we calculated. In the numerical example we even get a meaningful result for a relatively poor approximation.

The general philosophy of a-priori error analysis is to use the true solution, which is plugged into the numerical scheme to calculate the residual. Then using the discrete in time equation given by the numerical scheme, we can derive a discrete equation for the error, which has coefficients depending on the true solution. Using a-priori information of the solution, asymptotic bounds for the error are derived. In a-posteriori analysis we use a time-continuous interpolation of the numerical data, which is plugged into the SPDE, in order to derive the residual. For the error we obtain a PDE which is continuous in time and has coefficients depending on the numerical data. Here we can use now standard a-priori SPDE-type methods to derive error bounds, that depend only on numerical data and the residual.

Although for simplicity of presentation, we use a much simpler equation of Allen-Cahn-type, our result is motivated by equations where the global existence of solutions is not known, and thus global a-priori estimates are not available. Typical examples are the three-dimensional Navier-Stokes equation or a somewhat simpler equation from surface growth [7]. For the latter in [6, 16] a-posteriori analysis was used for the deterministic PDEs to prove numerically the regularity of solutions and thus the global existence and uniqueness.

Here we focus as a starting point for simplicity on an one-dimensional equation of Allen-Cahn type. Here even the asymptotic convergence results of numerical schemes are well known. See for example [15, 14] or [4] for a truncated scheme. Moreover, there is no problem with existence and uniqueness of solutions. See for example [9].

For the spatial discretization we use the spectral Galerkin-scheme, which simplifies the analysis. Moreover, for the time-discretization we use a variant of the exponential scheme introduced by [12]. Asymptotically, both results should be equivalent, but the variant we use is slightly easier in the analysis.

The precise functional analytic set-up and the equation itself is presented in Section 2. In Section 3 we present analytic results for stochastic terms we cannot evaluate numerically. One is the infinite-dimensional remainder of the stochastic convolution at discretization times. The second one bounds fluctuations in between discretization times. Here we need to analyze an Ornstein-Uhlenbeck bridge-process, as we now the stochastic convolution at all discretization times, both in the future and the past.

In the main result we present in Section 4 analytic error estimates for the residual that depend only on the numerically calculated data, the initial condition, and the stochastic terms already bounded in Section 3. We

provide a bound in moments of the L^4 -norm, which is conditioned on the given numerical data. In Section 5 we study the conditional mean square error of the approximation in L^2 -norm, given the numerical data. Nevertheless, this is a property of the equation and not of the data. We need to quantify the continuous dependence of solutions on additive perturbations like the stochastic convolution. Due to the relatively simple structure of the equation with a stable nonlinearity and a stable linear part, this is relatively straightforward.

In the final section, we give numerical examples to illustrate the result. Here we use a quite poor discretization given that the solution is very rough and still obtain meaningful error bounds. The main source of error comes from bounds on term that appear between the discretization points and not at the discrete times where the approximation is calculated.

2 Setting

The following assumptions and definitions are used throughout the paper. Consider the following SPDE on the Hilbert-space $H = L^2([0, \pi])$ which is of the type:

$$du = [Au + F(u)]dt + dW \quad u(0) = u_\star . \quad (1)$$

subject to Dirichlet boundary conditions on $[0, \pi]$, where A is the Laplacian, W some cylindrical Q -Wiener process. Finally, F is the locally-Lipschitz nonlinearity $F(u) = -u^3$.

The Dirichlet Laplacian A is diagonal w.r.t. $e_k(x) = \sqrt{2/\pi} \sin(kx)$, $k \in \mathbb{N}$ and generates an analytic semigroup $\{e^{tA}\}_{t \geq 0}$ on H . Moreover, it is a contraction semigroup on any $L^p(0, \pi)$. This follows in L^2 as the largest eigenvalue of A is -1 and thus $\|e^{tA}\|_{\mathcal{L}(L^2)} \leq e^{-t}$. In L^∞ it is true by the maximum principle $\|e^{tA}\|_{\mathcal{L}(L^\infty)} \leq 1$. Then by the Riesz-Thorin theorem for any L^p -space we have for $t > 0$

$$\|e^{tA}\|_{\mathcal{L}(L^p)} \leq \|e^{tA}\|_{\mathcal{L}(L^\infty)}^{(p-2)/p} \|e^{tA}\|_{\mathcal{L}(L^2)}^{2/p} \leq e^{-2t/p} < 1 . \quad (2)$$

Let us remark that by geometric series $I - e^{tA}$ is an invertible operator in L^p with bounded inverse.

Moreover, for simplicity we assume that the covariance operator Q is also diagonal in the Fourier basis e_k , and denote the eigenvalues by α_k^2 , i.e. $Qe_k = \alpha_k^2 e_k$. In the numerical examples we consider space-time-white noise of order one that corresponds to $\alpha_k = 1$ for all $k \in \mathbb{N}$, which is a case of solutions of quite poor regularity with strong fluctuations. Nevertheless we could allow for even rougher noise.

We assume that $\sum_{k \in \mathbb{N}} \alpha_k^2 k^{-2+\delta} < \infty$ for some $\delta > 0$, which guarantees that the stochastic convolution (or Ornstein-Uhlenbeck process) Z

$$Z(t) = \int_0^t e^{(t-s)A} dW(s) .$$

is continuous both in space and time (cf. [9]).

The mild formulation of (1) is defined by the fixed point equation

$$u(t) = e^{At}u_\star + \int_0^t e^{A(t-s)}F(u(s))ds + Z(t) . \quad (3)$$

Here the existence and uniqueness of solutions is standard. See for example [9]. Here we just need for some $p \geq 3$ that $u \in C^0([0, T], L^p)$, \mathbb{P} -almost sure, in order to formulate the mild solution in L^p and apply fixed point theorem to (3). For the rest of the paper we just assume that u_\star is such that there is a sufficiently smooth unique mild solution.

2.1 Discretization

Here we define the discretization scheme used throughout the paper. For the discretization in time, we use a fixed step-size $h = T/M > 0$ and for a fixed realization ω , using a random number generator, we obtain in principle exact values of

$$\{P_N Z(hk)\}_{k \in \mathbb{N}} ,$$

defined by

$$Z_0 = 0 , \quad Z_{k+1} = e^{hA}Z_k + X_{k+1} = \sum_{j=1}^{k+1} e^{h(k+1-j)A}P_N X_j$$

with independent and identically distributed \mathbb{R}^N -valued Gaussian random variables

$$X_{k+1} = P_N \int_{hk}^{h(k+1)} e^{(h(k+1)-s)A} dW(s) \sim \mathcal{N}(0, P_N \int_0^h e^{2sA} ds P_N) .$$

Given these values Z_k for $Z(hk)$, the numerical method provides a realization of the approximation

$$\{u_k\}_{k=0, \dots, M} \subset P_N H ,$$

which is defined recursively as $u_0 = P_N u_\star$ and

$$u_n = e^{Ah}P_N u_{n-1} + \int_0^h e^{A(h-s)}ds F_N(u_{n-1}) + X_n .$$

We can also write this explicitly as

$$u_n = e^{Anh}P_N u_0 + \sum_{k=1}^n \int_{(k-1)h}^{kh} e^{A(kh-s)}ds F_N(u_{k-1}) + Z_n .$$

Moreover, we define the approximation $\varphi : [0, T] \rightarrow P_N H$ by the linear interpolation of the points $\varphi(hk) = u_k$.

2.2 Result

The aim of this paper is to bound the conditional mean-square error given the numerical data, i.e, we want to obtain

$$\mathbb{E}[\|u - \varphi\|^2 \mid \{X_k\}_{k \in \mathbb{N}}] = \mathbb{E}[\|u - \varphi\|^2 \mid \{Z_k\}_{k \in \mathbb{N}}] = \text{small} ,$$

which is not an asymptotic result, but one that holds for the given approximation. In Theorem 13 we present the main analytic result for this statement. The term “small” depends on one hand on the numerical data, and we will evaluate this part only numerically. On the other hand, we have infinite-dimensional parts and random fluctuations between discretization points, which we have to bound analytically, as there is no data available.

The general philosophy is to evaluate as much as possible of the error bounds using the numerical data, and only rely on analytic estimates if no numerical evaluation is possible.

3 Stochastic bounds

Here we present analytic results for stochastic terms we cannot evaluate numerically. One is the infinite-dimensional reminder of the stochastic convolution at discretization times. The other one arises from fluctuations in between discretization times, where we need to analyze an Ornstein-Uhlenbeck bridge-process.

3.1 OU-process

For the stochastic term $Q_N Z(nh)$ we cannot use any numerical data to evaluate it. Moreover it is infinite dimensional. The main result here is Lemma 3 below. First we need estimates of a Gaussian in the L^4 -norm using the expansion in Fourier-series.

Lemma 1. *Let $Z \sim \mathcal{N}(0, Q)$ with a covariance-operator Q on H such that $\text{tr}(Q) < \infty$. Denote the eigenvalues and eigenfunctions by $Qe_k = \alpha_k^2 e_k$ and suppose that for all x we have $\sum_{k \in \mathbb{N}} \alpha_k^2 e_k^2(x) < \infty$, then*

$$\mathbb{E}\|Z\|_{L^4}^4 = 3 \sum_{k, \ell} \alpha_k^2 \alpha_\ell^2 \int e_k^2(x) e_\ell^2(x) dx .$$

Proof. By assumption, we obtain that for all x the real-valued random variable $Z(x) \sim \mathcal{N}(0, \sum_{k \in \mathbb{N}} \alpha_k^2 e_k^2(x))$. Thus by Tonelli’s theorem

$$\mathbb{E}\|Z\|_{L^4}^4 = \int \mathbb{E}|Z(x)|^4 dx = 3 \int (\mathbb{E}|Z(x)|^2)^2 dx = 3 \int \left(\sum_k \alpha_k^2 e_k^2(x) \right)^2 dx$$

which implies the claim. Note that we used that we can explicitly calculate the fourth moment of a centered scalar Gaussian. \square

Recall that the Fourier-basis $\{e_k\}_{k \in \mathbb{N}}$ is defined by

$$e_k(x) = \sqrt{2} \sin(kx) / \sqrt{\pi}.$$

Here we have for $k = 1, 2, \dots$ where $f_k(x) = \sqrt{2} \cos(kx) / \sqrt{\pi}$

$$e_k^2(x) = \frac{2}{\pi} \sin^2(kx) = \frac{1}{\pi} - \frac{1}{\sqrt{2\pi}} f_{2k}(x).$$

Thus we have for $k, \ell > 0$

$$\int_0^\pi e_k^2(x) e_\ell^2(x) dx = \frac{1}{\pi} + \frac{1}{2\pi} \delta_{k,\ell}.$$

Now we can verify

$$\begin{aligned} \int_0^\pi \sum_{k,\ell} \alpha_k^2 \alpha_\ell^2 e_k^2(x) e_\ell^2(x) dx &= \frac{1}{\pi} \left(\sum_k \alpha_k^2 \right)^2 + \frac{1}{2\pi} \sum_k \alpha_k^4 \\ &\leq \frac{3}{2\pi} \left(\sum_k \alpha_k^2 \right)^2. \end{aligned}$$

This yields the following lemma:

Lemma 2. *Let $Z \sim \mathcal{N}(0, Q)$ with a covariance-operator Q on H such that $\text{tr}(Q) < \infty$. Let Q be diagonal w.r.t. the Fourier-basis e_k , then*

$$\mathbb{E} \|Z\|_{L^4}^4 \leq \frac{3}{2\pi} (\text{tr}(Q))^2 = \frac{3}{2\pi} (\mathbb{E} \|Z\|_{L^2}^2)^2.$$

We furthermore obtain:

Lemma 3. *The sequence $\{Q_N Z(kh)\}_{k=1,\dots,M}$ is independent of $(Z_k)_{k \in \mathbb{N}}$ and bounded in L^4 -norm for all $N \in \mathbb{N}$ by*

$$\sup_{t \geq 0} \mathbb{E} \|Q_N Z(t)\|_{L^4}^4 \leq \frac{3}{8\pi} \left(\sum_{k > N} \alpha_k^2 k^{-2} \right)^2.$$

A stronger result as proven in [5], where we even could take the supremum in time over bounded intervals inside the expectation and use L^∞ instead of L^4 -norms.

The bound of Lemma 3 is still not numerically computable, but given a bounded sequence α_k , it is usually straightforward to evaluate (or bound) the series explicitly. See Remark 11.

Proof. We start by using Lemma 2

$$\begin{aligned} \mathbb{E} \|Q_N Z(t)\|_{L^4}^4 &\leq \frac{3}{2\pi} (\mathbb{E} \|Q_N Z(t)\|_{L^2}^2)^2 \\ &= \frac{3}{2\pi} \left(\sum_{k > N} \alpha_k^2 \int_0^t e^{-2k^2 s} ds \right)^2. \end{aligned}$$

This easily implies the claim. □

3.2 OU-bridge

In order to treat random fluctuations between discretization points, we define for $\tau \in (0, h)$

$$\mathcal{Z}_n(\tau) = \int_{nh}^{nh+\tau} e^{A(nh+\tau-s)} dW_s. \quad (4)$$

First the processes $\{\mathcal{Z}_n\}_{n \in \mathbb{N}}$ are independent and identically distributed. Denote also the high modes $\mathcal{Z}_n^{(h)} = (I - P_N)\mathcal{Z}_n$ and the low modes $\mathcal{Z}_n^{(l)} = P_N\mathcal{Z}_n$, which are by definition mutually independent.

Moreover, it is easy to see that $\{\mathcal{Z}_n(\tau)\}_{\tau \in [0, h]}$ depends on $\{Z_k\}_{k \in \mathbb{N}}$ only via $\mathcal{Z}_n^{(l)}(h) = Z_{n+1} - e^{hA}Z_n$. Thus, recalling that Z is a Markov-process we obtain

$$\mathbb{E}\left[\int_0^h \|\mathcal{Z}_n(\tau)\|_{L^4}^4 d\tau \mid (Z_k)_{k \in \mathbb{N}}\right] = \mathbb{E}\left[\int_0^h \|\mathcal{Z}_n(\tau)\|_{L^4}^4 d\tau \mid \mathcal{Z}_n^{(l)}(h)\right].$$

We obtain the following Lemma:

Lemma 4.

$$\mathbb{E}\left[\int_0^h \|\mathcal{Z}_n(\tau)\|_{L^4}^4 d\tau \mid (Z_k)_{k \in \mathbb{N}}\right] = \mathcal{N}(Z_{n+1} - e^{hA}Z_n)$$

where

$$\mathcal{N}(z) = \mathbb{E}\left[\int_0^h \|\mathcal{Z}_0(\tau)\|_{L^4}^4 d\tau \mid \mathcal{Z}_0^{(l)}(h) = z\right]$$

Note that this splits into the infinite dimensional remainder and an OU-bridge process on the low modes. For the OU-bridge process by a result of [11] we know explicitly the law:

Lemma 5. For $t \in [0, h]$ the law of $\mathcal{Z}_0^{(l)}(t)$ given $\mathcal{Z}_0^{(l)}(h) = z$ with $z \in P_N H$ is a Gaussian with mean $\lambda(t, z)$ and covariance \tilde{Q}_t with

$$\lambda(t, z) = 2P_N \int_0^t e^{2As} ds \cdot e^{A(h-t)} [I - e^{2hA}]^{-1} Az$$

and

$$\tilde{Q}_t = P_N \frac{Q}{2A} \frac{1 - e^{2tA}}{1 - e^{2hA}} \left(I - e^{2A(h-t)} \right).$$

Proof. We follow the result of [11], but our setting is much simpler. First all operators involved are diagonal and thus symmetric. Furthermore, they all commute. We can also treat degenerate noise by restricting the results of [11] to the Hilbert-space given by the range of Q , which is in general only a subset of $P_N H$. But then both Q and A are invertible on that space.

First,

$$\text{Law}[\mathcal{Z}_0^{(l)}(t)|\mathcal{Z}_0^{(l)}(h) = z] = \mathcal{N}(\lambda(t, z), \tilde{Q}_t)$$

where by [11, Prop. 2.11]

$$\tilde{Q}_t = P_N Q_t (I - V_t^2)$$

and by [11, Prop. 2.13]

$$\lambda(t, z) = P_N K_t Q_h^{-1/2} z$$

with the following operators,

$$Q_t = Q \int_0^t e^{2As} ds, \quad V_t = Q_h^{-\frac{1}{2}} e^{A(h-t)} Q_t^{\frac{1}{2}}, \quad K_t = Q_t^{\frac{1}{2}} V_t = Q_h^{-\frac{1}{2}} e^{A(h-t)} Q_t.$$

This implies

$$\lambda(t, z) = P_N Q_h^{-1} e^{A(h-t)} Q_t z = P_N \int_0^t e^{2As} ds \cdot e^{A(h-t)} \left[\int_0^h e^{2As} ds \right]^{-1} z.$$

and

$$\begin{aligned} \tilde{Q}_t &= P_N Q_t (I - Q_h^{-1} e^{2A(h-t)} Q_t) \\ &= P_N Q_t Q_h^{-1} (Q_h - e^{2A(h-t)} Q_t) \\ &= P_N Q \frac{1 - e^{2tA}}{1 - e^{2hA}} \frac{1}{2} A^{-1} \left(I - e^{2hA} - e^{2A(h-t)} (1 - e^{2tA}) \right) \\ &= P_N \frac{Q}{2A} \frac{1 - e^{2tA}}{1 - e^{2hA}} \left(I - e^{2A(h-t)} \right). \end{aligned}$$

□

The following bound is surely not optimal, but a slight simplification of the exact bound.

Lemma 6. *We bound for $z \in P_N$*

$$\mathcal{N}(z) \leq h \cdot \mathcal{S}_h(z)$$

where we define

$$\mathcal{S}_h(z) = \left[\frac{2h}{\sqrt[4]{5}} \| [I - e^{2hA}]^{-1} A z \|_{L^4} + \left(\frac{3}{2\pi} \right)^{1/4} \left(h^{-1/2} \Sigma_N(h) + \sum_{k=N+1}^{\infty} \frac{\alpha_k^2}{2k^2} \right)^{1/2} \right]^4$$

where

$$\Sigma_N(h) = \sum_{k=1}^N \frac{\alpha_k^2}{\sqrt{8} \cdot k^3} \frac{(8hk^2 e^{-2hk^2} + (2hk^2 + 3)e^{-4hk^2} + 2hk^2 - 3)^{1/2}}{1 - e^{-2hk^2}}.$$

We can explicitly calculate an upper bound for \mathcal{S}_h by using numerical data for z and for the infinite series we can use

$$\sum_{k>N} \frac{\alpha_k^2}{k^2} \leq \frac{\sup_{k>N} \alpha_k^2}{N}.$$

Proof. First using Lemmas 5 and 2 and taking into account the infinite-dimensional remainder of the OU-process, that is independent of the OU-bridge, we obtain

$$\text{Law}[\mathcal{Z}_0(t)|\mathcal{Z}_0^{(l)}(h) = z] = \mathcal{N}(\lambda(t, z), \tilde{\mathcal{Q}}_t)$$

on H with covariance operator $\hat{\mathcal{Q}}_t$ being diagonal in Fourier space with

$$P_N \hat{\mathcal{Q}}_t = \tilde{\mathcal{Q}}_t \quad \text{and} \quad (I - P_N) \hat{\mathcal{Q}}_t = Q \int_0^t e^{2sA} ds$$

where

$$\mathcal{N}(z)^{1/4} \leq \left(\int_0^h \|\lambda(s, z)\|_{L^4}^4 ds \right)^{1/4} + \left(\frac{3}{2\pi} \int_0^h \text{trace}(\tilde{\mathcal{Q}}_s)^2 ds \right)^{1/4}$$

Now

$$\begin{aligned} \text{trace}(\hat{\mathcal{Q}}_s) &= \text{trace}(\tilde{\mathcal{Q}}_s) + \text{trace}\left(Q \int_0^t e^{2\eta A} d\eta\right) \\ &= \sum_{k=1}^N \frac{\alpha_k^2}{2k^2} \frac{1 - e^{-2sk^2}}{1 - e^{-2hk^2}} \left(1 - e^{-2k^2(h-s)}\right) + \sum_{k=N+1}^{\infty} \frac{\alpha_k^2}{2k^2} \left(1 - e^{-2k^2s}\right) \end{aligned}$$

and thus

$$\begin{aligned} \int_0^h \text{trace}(\hat{\mathcal{Q}}_s)^2 ds &= \|\text{trace}(\hat{\mathcal{Q}}_s)\|_{L^2(0,h)}^2 \\ &\leq \left(\|\text{trace}(\tilde{\mathcal{Q}}_s)\|_{L^2(0,h)} + \|\text{trace}(Q \int_0^t e^{2\eta A} d\eta)\|_{L^2(0,h)} \right)^2 \end{aligned}$$

where we bound

$$\begin{aligned} &\|\text{trace}(\tilde{\mathcal{Q}}_s)\|_{L^2(0,h)} \\ &\leq \sum_{k=1}^N \frac{\alpha_k^2}{2k^2} \frac{1}{1 - e^{-2hk^2}} \|(1 - e^{-2sk^2})(1 - e^{-2k^2(h-s)})\|_{L^2(0,h)} \\ &\leq \sum_{k \in \mathbb{N}} \frac{\alpha_k^2}{2k^2} \frac{1}{1 - e^{-2hk^2}} \frac{1}{\sqrt{2k^2}} \left(8hk^2 e^{-2hk^2} + (2hk^2 + 3)e^{-4hk^2} + 2hk^2 - 3\right)^{1/2} \\ &\leq \sum_{k \in \mathbb{N}} \frac{\alpha_k^2}{\sqrt{8} \cdot k^3} \frac{(8hk^2 e^{-2hk^2} + (2hk^2 + 3)e^{-4hk^2} + 2hk^2 - 3)^{1/2}}{1 - e^{-2hk^2}} = \Sigma_N(h). \end{aligned}$$

Moreover,

$$\text{trace}(Q \int_0^\cdot e^{2\eta A} d\eta) \|_{L^2(0,h)} \leq \sum_{k=N+1}^{\infty} \frac{\alpha_k^2}{2k^2} \|1 - e^{-2k^2 t}\|_{L^2(0,h)} \leq h \sum_{k=N+1}^{\infty} \frac{\alpha_k^2}{2k^2}.$$

Thus

$$\int_0^h \text{trace}(\tilde{Q}_s)^2 ds \leq \left(\Sigma_N(h) + h \sum_{k=N+1}^{\infty} \frac{\alpha_k^2}{2k^2} \right)^2$$

For the mean value, we obtain from Lemma 5 using (2)

$$\|\lambda(t, z)\|_{L^4} \leq 2t \| [I - e^{2hA}]^{-1} Az \|_{L^4}.$$

Thus

$$\int_0^h \|\lambda(t, z)\|_{L^4}^4 dt \leq \frac{2^4}{5} h^5 \| [I - e^{2hA}]^{-1} Az \|_{L^4}^4.$$

□

4 Residual estimates

This section is devoted bounds on the residual, which measures the quality of an arbitrary numerical approximation. First we consider the discretization points $t_n = nh$, and later we focus on the points $nh + \tau$, $\tau \in (0, h)$, which are in between.

Definition 7. For the approximation $\varphi : [0, T] \rightarrow P_N H$ defined in Section 2.1 and $t \in [0, T]$ we define the residual

$$Res(t) = \varphi(t) - e^{At} \varphi(0) - \int_0^t e^{A(t-s)} F(\varphi(s)) ds - Z(t). \quad (5)$$

4.1 At discretization points

At first for the points $t_n = nh$ we have

$$Res(nh) = - \sum_{k=1}^n \int_{(k-1)h}^{kh} e^{A(nh-s)} (F(\varphi(s)) - P_N F(u_{k-1})) ds - Q_N Z(nh).$$

In the following Lemma we estimate the Residual at the discretization points nh using the numerical data.

Lemma 8. The residual $\{Res(kh) : k = 0, \dots, M\}$ defined in (5) at discrete times is given as

$$Res(kh) = Res_k^{dat} + Res_k^{stoch}$$

where

$$Res_k^{dat} = I_2(k) + I_3(k)$$

given by the recursive schema $I_j(0) = 0$,

$$I_2(n) = e^{hA} I_2(n-1) + hQ_N \int_0^1 e^{sAh} \left[u_n^3 - s^3(u_n)^2 d_n + s^2 3u_n(d_n)^2 - s^3(d_n)^3 \right] ds$$

and

$$I_3(n) = e^{hA} I_3(n-1) + hP_N \int_0^1 e^{sAh} \left[(u_n^3 - u_{n-1}^3) - 3s d_n u_n^2 + 3s^2 d_n^2 u_n - s^3 d_n^3 \right] ds.$$

Moreover,

$$Res_n^{stoch} = -Q_N Z(nh)$$

is random and independent of the numerical data.

The value of Res^{dat} just depends on the numerical data u_k and $d_k = u_k - u_{k-1}$. Note that the cubic terms depending on u_k and d_k are all in H_{3N} and thus computable. The integrals are all diagonal matrices that can be calculated explicitly in the numerical evaluation.

Proof. We have

$$\begin{aligned} Res(nh) &= - \sum_{k=1}^n \int_{(k-1)h}^{kh} e^{A(nh-s)} Q_N F(\varphi(s)) ds \\ &\quad - \sum_{k=1}^n \int_{(k-1)h}^{kh} e^{A(nh-s)} P_N [(F(\varphi(s)) - F(u_{k-1}))] d\tau \quad (6) \\ &\quad - Q_N Z(nh) \\ &=: I_2(n) + I_3(n) - Q_N Z(nh). \end{aligned}$$

For the two integrals we use for $s \in [(k-1)h, kh]$ that $\varphi(s) = u_{k-1} + \frac{\tau}{h} d_k$, where $s = (k-1)h + \tau$ with $\tau \in (0, h)$ and $d_k = u_k - u_{k-1}$, which depends only on the numerical data.

For the first integral I_2 on the right hand side we note that although it looks infinite dimensional, due to the cubic nonlinearity it is finite dimen-

sional and we can calculate it explicitly

$$\begin{aligned}
I_2(n) &:= - \sum_{k=1}^n \int_0^h e^{A(h(n-k+1)-s)} Q_N F(u_{k-1} + \frac{s}{h} d_k) ds \\
&= -h \sum_{k=1}^n \int_0^1 e^{Ah(n-k+s)} Q_N F(u_k - s d_k) ds \\
&= e^{hA} I_2(n-1) - h \int_0^1 e^{Ahs} Q_N F(u_k - s d_k) ds \\
&= e^{hA} I_2(n-1) + h Q_N \left[\int_0^1 e^{Ahs} ds \cdot u_n^3 - \int_0^1 e^{Ahs} s ds \cdot 3(u_n)^2 d_n \right] \\
&\quad + h Q_N \left[\int_0^1 e^{Ahs} s^2 ds \cdot 3u_n(d_n)^2 - \int_0^1 e^{Ahs} s^3 ds \cdot (d_n)^3 \right]
\end{aligned}$$

For the next integral I_3 in (6) we can proceed similarly as for I_2

$$\begin{aligned}
I_3(n) &= \sum_{k=1}^n \int_{(k-1)h}^{kh} e^{A(nh-s)} P_N [\varphi(s)^3 - (u_{k-1})^3] ds \\
&= \sum_{k=1}^n \int_{(k-1)h}^{kh} e^{A(nh-s)} P_N \left[(u_{k-1} + \frac{s - (k-1)h}{h} d_k)^3 - (u_{k-1})^3 \right] ds \\
&= h \sum_{k=1}^n \int_0^1 e^{Ah(n-k+1-s)} P_N [(u_{k-1} + s d_k)^3 - (u_{k-1})^3] ds \\
&= h \sum_{k=1}^n \int_0^1 e^{Ah(n-k+s)} P_N [(u_{k-1} + (1-s) d_k)^3 - (u_{k-1})^3] ds \\
&= h \sum_{k=1}^n \int_0^1 e^{Ah(n-k+s)} P_N [(u_k - s d_k)^3 - (u_{k-1})^3] ds \\
&= e^{hA} I_3(n-1) + h \int_0^1 e^{sAh} P_N [(u_n - s d_n)^3 - (u_{n-1})^3] ds
\end{aligned}$$

By expanding the cubic we have

$$\begin{aligned}
I_3(n) &= e^{hA} I_3(n-1) + h \int_0^1 e^{sAh} P_N ds \cdot (u_n^3 - u_{n-1}^3) - 3h \int_0^1 e^{sAh} s ds P_N \cdot d_n u_n^2 \\
&\quad + 3h \int_0^1 e^{sAh} P_N s^2 ds \cdot d_n^2 u_n - h \int_0^1 e^{sAh} P_N s^3 ds \cdot d_n^3.
\end{aligned}$$

□

4.2 General case

For the residual at times between the numerical grid points we have for $\tau \in (0, h)$

$$\begin{aligned}
& -\text{Res}(nh + \tau) \\
&= e^{A(nh+\tau)}\varphi(0) + \int_0^{nh+\tau} e^{A(nh+\tau-s)}F(\varphi(s))ds + Z(nh + \tau) - \varphi(nh + \tau) \\
&= e^{A(nh+\tau)}\varphi(0) + \int_0^{nh} e^{A(nh+\tau-s)}F(\varphi(s))ds + \int_{nh}^{nh+\tau} e^{A(nh+\tau-s)}F(\varphi(s))ds \\
&\quad + e^{A\tau}Z_n + \int_{nh}^{nh+\tau} e^{A(nh+\tau-s)}dW_s - \varphi(nh + \tau) \\
&= e^{A\tau}\left[e^{nhA}\varphi(0) + \int_0^{nh} e^{A(nh-s)}F(\varphi(s))ds + \int_{nh}^{nh+\tau} e^{A(nh-s)}F(\varphi(s))ds + Z_n\right] \\
&\quad + \int_{nh}^{nh+\tau} e^{A(nh+\tau-s)}dW_s - \varphi(nh + \tau) \\
&= e^{A\tau}\left[\text{Res}(nh) + u_n + \int_{nh}^{nh+\tau} e^{A(nh-s)}F(\varphi(s))ds\right] \\
&\quad + \int_{nh}^{nh+\tau} e^{A(nh+\tau-s)}dW_s - \varphi(nh + \tau).
\end{aligned} \tag{7}$$

Therefore by the fact that by linear interpolation $\varphi(nh + \tau) = u_{nh} + \frac{\tau}{h}(d_{n+1})$, where $d_{n+1} = u_{n+1} - u_n$ we get

$$\begin{aligned}
\text{Res}(nh + \tau) &= e^{A\tau}\text{Res}(nh) + (e^{A\tau} - I)u_n - \frac{\tau}{h}d_{n+1} \\
&\quad + \int_{nh}^{nh+\tau} e^{A(nh+\tau-s)}F(\varphi(s))ds + \int_{nh}^{nh+\tau} e^{A(nh+\tau-s)}dW_s.
\end{aligned} \tag{8}$$

Now at some point we need to estimate, as due to the τ , we cannot evaluate the terms numerically explicit.

For the first integral term in the right hand side of (8) we have

$$I(\tau) = \int_{nh}^{nh+\tau} e^{A(nh+\tau-s)}F(\varphi(s))ds = -h \int_0^{\frac{\tau}{h}} e^{A(\tau-sh)}(u_n + sd_{n+1})^3ds. \tag{9}$$

In order to bound $I(\tau)$, we use that the semi-group e^{tA} generated by the Dirichlet-Laplacian A is a contraction semigroup on any $L^p(0, \pi)$. See (2). Thus we obtain

$$\begin{aligned}
\|I(\tau)\|_{L^4} &\leq h \int_0^{\frac{\tau}{h}} \|(u_n + s d_{n+1})^3\|_{L^4} ds \\
&\leq h \int_0^{\frac{\tau}{h}} (\|u_n\|_{L^{12}} + s \|d_{n+1}\|_{L^{12}})^3 ds \\
&\leq \tau \|u_n\|_{L^{12}}^3 + \frac{3}{2} \frac{\tau^2}{h} \|u_n\|_{L^{12}}^2 \|d_{n+1}\|_{L^{12}} \\
&\quad + \frac{\tau^3}{h^2} \|u_n\|_{L^{12}} \|d_{n+1}\|_{L^{12}}^2 + \frac{1}{4} \frac{\tau^4}{h^3} \|d_{n+1}\|_{L^{12}}^3 .
\end{aligned} \tag{10}$$

This still contains powers of τ , but as we are going to integrate this over τ , we keep them and estimate later. Let us summarize the result starting from (8).

Lemma 9. *For $n \in \{0, \dots, M-1\}$ and $\tau \in [0, h]$ we have*

$$Res(nh + \tau) = e^{A\tau} Res(nh) + (e^{A\tau} - I)u_n - \frac{\tau}{h} d_{n+1} + I(\tau) + \mathcal{Z}_n(\tau). \tag{11}$$

with I defined in (9) and bounded in (10) and \mathcal{Z}_n was defined in (4).

4.3 Bounding the residual

Now we are going to bound the residual for intermediate times. Let us fix $n \in \{0, \dots, M-1\}$ and $\tau \in [0, h]$. In view of Lemma 9. We first bound

$$\|(e^{A\tau} - I)u_n\|_{L^4} \leq \left\| \int_0^\tau A e^{As} ds u_n \right\|_{L^4} \leq \tau \|A u_n\|_{L^4}$$

to obtain

$$\|Res(nh + \tau)\|_{L^4} \leq \|Res(nh)\|_{L^4} + \tau \|A u_n\|_{L^4} + \frac{\tau}{h} \|d_{n+1}\|_{L^4} + \|I(\tau)\|_{L^4} + \|\mathcal{Z}_n(\tau)\|_{L^4}. \tag{12}$$

In order to bound the residual in a conditional L^4 -moment, we define

$$n(s) = n \iff s \in [nh, h(n+1)) \quad \text{and} \quad \tau(s) = s - n(s)h \in [0, h) .$$

We fix $t \in [0, T]$ and obtain from (12) by triangle inequality

$$\begin{aligned}
& \left(\mathbb{E} \left[\int_0^t \|\text{Res}(\tau)\|_{L^4}^4 d\tau \mid (Z_k)_{k \in \mathbb{N}} \right] \right)^{1/4} \\
&= \left(\mathbb{E} \left[\int_0^t \|\text{Res}(n(s)h + \tau(s))\|_{L^4}^4 ds \mid (Z_k)_{k \in \mathbb{N}} \right] \right)^{1/4} \\
&\leq \left(\mathbb{E} \left[\int_0^t \|\text{Res}(n(s)h)\|_{L^4}^4 ds \mid (Z_k)_{k \in \mathbb{N}} \right] \right)^{1/4} + \left(\int_0^t \tau(s)^4 \|Au_{n(s)}\|_{L^4}^4 ds \right)^{1/4} \\
&+ \left(\int_0^t \frac{\tau(s)^4}{h^4} \|d_{n(s)+1}\|_{L^4}^4 ds \right)^{1/4} + \left(\int_0^t \|I(\tau(s))\|_{L^4}^4 ds \right)^{1/4} \\
&+ \left(\mathbb{E} \left[\int_0^t \|Z_{n(s)}(\tau(s))\|_{L^4}^4 ds \mid (Z_k)_{k \in \mathbb{N}} \right] \right)^{1/4}.
\end{aligned} \tag{13}$$

We now bound all the terms above separately. Let $m(t)$ be the largest integer, such that $m(t)h \leq t$. From Lemma 8 again by triangle inequality, we get

$$\begin{aligned}
& \left(\mathbb{E} \left[\int_0^t \|\text{Res}(n(s)h)\|_{L^4}^4 ds \mid (Z_k)_{k \in \mathbb{N}} \right] \right)^{1/4} \\
&\leq \left(\int_0^t \|\text{Res}_{n(s)}^{\text{dat}}\|_{L^4}^4 ds \right)^{1/4} + \left(\mathbb{E} \int_0^t \|\text{Res}_{n(s)}^{\text{stoch}}\|_{L^4}^4 ds \right)^{1/4} \\
&\leq \left(h \sum_{n=1}^{m(t)} \|\text{Res}_n^{\text{dat}}\|_{L^4}^4 \right)^{1/4} + \left(h \sum_{n=1}^{m(t)} \mathbb{E} \|\text{Res}_n^{\text{stoch}}\|_{L^4}^4 \right)^{1/4} \\
&\leq \left(h \sum_{n=1}^{m(t)} \|\text{Res}_n^{\text{dat}}\|_{L^4}^4 \right)^{1/4} + \left(\frac{3hm(t)}{8\pi} \right)^{1/4} \left(\sum_{k>N} \frac{\alpha_k^2}{k^2} \right)^{1/2}
\end{aligned}$$

where we used that $\text{Res}(0) = 0$, so that all sums start at $n = 1$.

For the next term

$$\begin{aligned}
\left(\int_0^t \tau(s)^4 \|Au_{n(s)}\|_{L^4}^4 ds \right)^{1/4} &\leq \left(\sum_{n=0}^{m(t)} \int_{nh}^{(n+1)h} \tau(s)^4 ds \|Au_n\|_{L^4}^4 ds \right)^{1/4} \\
&\leq \left(\sum_{n=0}^{m(t)} \int_0^h s^4 ds \|Au_n\|_{L^4}^4 ds \right)^{1/4} \\
&\leq \left(\frac{h^5}{5} \sum_{n=0}^{m(t)} \|Au_n\|_{L^4}^4 ds \right)^{1/4}
\end{aligned}$$

and similarly

$$\left(\int_0^t \frac{\tau(s)^4}{h^4} \|d_{n(s)+1}\|_{L^4}^4 ds \right)^{1/4} \leq \left(\frac{h}{5} \sum_{n=0}^{m(t)} \|d_{n+1}\|_{L^4}^4 ds \right)^{1/4}$$

For the integral-term I by (10)

$$\left(\int_0^t \|I(\tau(s))\|_{L^4}^4 ds \right)^{1/4} \leq \left(h^2 \sum_{n=0}^{m(t)} \left[\frac{1}{2} \|u_n\|_{L^{12}}^3 + \frac{1}{2} \|u_n\|_{L^{12}}^2 \|d_{n+1}\|_{L^{12}} \right. \right. \\ \left. \left. + \frac{1}{4} \|u_n\|_{L^{12}} \|d_{n+1}\|_{L^{12}}^2 + \frac{1}{20} \|d_{n+1}\|_{L^{12}}^3 \right] \right)^{1/4}.$$

Finally for the Ornstein-Uhlenbeck bridge, we have from Lemma 6

$$\left(\mathbb{E} \left[\int_0^t \|Z_{n(s)}(\tau(s))\|_{L^4}^4 ds \mid (Z_k)_{k \in \mathbb{N}} \right] \right)^{1/4} \leq \left(h \sum_{n=0}^{m(t)} \mathcal{S}_h(Z_{n+1} - e^{hA} Z_n) \right)^{1/4}.$$

Summarizing, we have the following bound:

Theorem 10. *With numerical data φ from Section 2.1 and the residual defined in (5) we have for $t \in [mh, (m+1)h)$ and $m \in \{1, \dots, M\}$ with $M = T/h$*

$$\mathbb{E} \left[\int_0^t \|Res(s)\|_{L^4}^4 ds \mid (Z_k)_{k \in \mathbb{N}} \right] \leq \mathcal{K}_m^4$$

with

$$\mathcal{K}_m = \left(h \sum_{n=1}^m \|Res_n^{dat}\|_{L^4}^4 \right)^{1/4} + \left(\frac{3mh}{8\pi} \right)^{1/4} \left(\sum_{k>N} \frac{\alpha_k^2}{k^2} \right)^{1/2} \\ + \left(\frac{h^5}{5} \sum_{n=0}^m \|Au_n\|_{L^4}^4 ds \right)^{1/4} + \left(h \sum_{n=0}^m \mathcal{S}_h(Z_{n+1} - e^{hA} Z_n) \right)^{1/4} \\ + \left(h^2 \sum_{n=0}^m \left[\frac{1}{2} \|u_n\|_{L^{12}}^3 + \frac{1}{2} \|u_n\|_{L^{12}}^2 \|d_{n+1}\|_{L^{12}} \right. \right. \\ \left. \left. + \frac{1}{4} \|u_n\|_{L^{12}} \|d_{n+1}\|_{L^{12}}^2 + \frac{1}{20} \|d_{n+1}\|_{L^{12}}^3 \right] \right)^{1/4}$$

Remark 11. *The quantity \mathcal{K}_m is almost numerically computable using numerical data. Moreover, we can update the sums in the numerical computation, so that we do not need to calculate them in every step.*

The only term that is not yet computable is the sum depending on the α_k for $k > N$, but here one can easily give an upper bound, once the α_k are given, by

$$\sum_{k>N} \frac{\alpha_k^2}{k^2} \leq \sup_{k>N} \{\alpha_k^2\} \int_N^\infty k^{-2} dk = \frac{1}{N} \sup_{k>N} \{\alpha_k^2\}.$$

Let us also remark that due to the way we did the estimate, we cannot take the number N of Fourier-modes arbitrarily large. Do to the regularity of the solution u , which is not in H^2 , we cannot expect $\|Au_n\|$ to be bounded for $N \rightarrow \infty$. Thus we always need to take h sufficiently small to balance that effect.

5 Approximating the error

In this section, we are mainly based on the properties of the equation. The numerical data only comes into play via the residual. We need to quantify the continuous dependence of solutions on additive perturbations like the stochastic convolution or the residual. Recall the mild solution of (1)

$$u(t) = e^{At}u_{\star} + \int_0^t e^{A(t-s)}F(u(s))ds + Z(t),$$

and the definition of the residual

$$\varphi(t) = e^{At}\varphi(0) + \int_0^t e^{A(t-s)}F(\varphi(s))ds + Z(t) + \text{Res}(t).$$

Therefore by putting $d(t) = u(t) - \varphi(t)$ we have

$$d(t) = u(t) - \varphi(t) = e^{tA}d(0) + \int_0^t e^{A(t-s)}(F(u(s)) - F(\varphi(s)))ds - \text{Res}(t)$$

with $d(0) = u_{\star} - \varphi(0) = Q_N u_{\star}$.

Substituting $r = d + \text{Res}$ we obtain

$$r(t) = e^{At}d(0) + \int_0^t e^{A(t-s)}[F(r(s) + \varphi(s) - \text{Res}(s)) - F(\varphi(s))]ds,$$

which means r is the solution of the following differential equation

$$\partial_t r = Ar + F(r + \varphi - \text{Res}) - F(\varphi).$$

Recall that $\text{Res}(0) = 0$ so $r(0) = d(0) = Q_N u_{\star}$.

Now we use standard a-priori estimates for the equation for r . This yields good estimates, as both the linear part and the nonlinear part are stable, which simplifies the error estimate significantly.

$$\frac{1}{2} \frac{\partial}{\partial t} \|r\|_{L_2}^2 = \langle Ar, r \rangle - \langle (r + \varphi - \text{Res})^3 - \varphi^3, r \rangle. \quad (14)$$

The following lemma is necessary to bound the cubic. It is not optimal, but sufficient for our purposes.

Lemma 12. *For all $r, R, \varphi \in \mathbb{R}$ we have*

$$[-(r + R + \varphi)^3 + \varphi^3]r \leq R^4 + 3R^2\varphi^2$$

Proof. First

$$[-(r + R + \varphi)^3 + \varphi^3]r = -3r \int_{\varphi}^{r+R+\varphi} \zeta^2 d\zeta = -3r \int_0^{r+R} (\zeta + \varphi)^2 d\zeta.$$

Thus the term is non-positive if r and $r + R$ have the same sign (i.e., in the case $r, r + R \in [0, \infty)$ or $r, r + R \in (-\infty, 0]$).

In the remaining two cases we have $|R| \leq |r|$, as for $r \leq 0 \leq R + r$ we have $R \geq -r \geq 0$ and for $r + R \leq 0 \leq r$ we have $0 \leq r \leq -R$. Thus we obtain using $ab \leq a^2 + \frac{1}{4}b^2$

$$\begin{aligned}
[-(r + R + \varphi)^3 + \varphi^3]r &= -r^4 - 3r^3(R + \varphi) - 3r^2(R + \varphi)^2 - r[(R + \varphi)^3 - \varphi^3] \\
&\leq -\frac{3}{4}r^2(R + \varphi)^2 - r[(R + \varphi)^3 - \varphi^3] \\
&= -\frac{3}{4}r^2(R + \varphi)^2 - rR^3 - 3rR\varphi(R + \varphi) \\
&\leq -rR^3 + 3R^2\varphi^2 \\
&\leq R^4 + 3R^2\varphi^2.
\end{aligned}$$

□

Now by Lemma 12 we obtain from (14)

$$\frac{\partial}{\partial t} \|r\|_{L^2}^2 \leq 2\|\text{Res}\|_{L^4}^4 + 6\|\text{Res}\|_{L^4}^2 \|\varphi\|_{L^4}^2,$$

and integration yields

$$\|r(t)\|_{L^2}^2 \leq \|r(0)\|_{L^2}^2 + 2 \int_0^t \|\text{Res}\|_{L^4}^4 ds + 6 \left(\int_0^t \|\text{Res}\|_{L^4}^4 ds \right)^{1/2} \left(\int_0^t \|\varphi\|_{L^4}^4 ds \right)^{1/2}$$

From Theorem 10 we can get the following bound for the error (by using Cauchy-Schwarz) in case $t \in [mh, (m+1)h]$

$$\mathbb{E}[\|r(t)\|_{L^2}^2 | (Z_k)_{k \in \mathbb{N}}] \leq \|Q_N u_\star\|^2 + 2(\mathcal{K}_m)^4 + 6(\mathcal{K}_m)^2 \left(\int_0^t \|\varphi\|_{L^4}^4 ds \right)^{1/2}$$

In order to have a fully numerically computable quantity, we need to take care of the integral. We proceed similarly to $I(\tau)$ and use $\varphi(t) = u_{n(t)} + \tau(t)h^{-1}d_{n(s)+1}$ to obtain

$$\begin{aligned}
\int_0^t \|\varphi\|_{L^4}^4 ds &\leq \int_0^t \|u_{n(s)} + \tau(s)h^{-1}d_{n(s)+1}\|_{L^4}^4 ds \\
&= \sum_{n=0}^{m(t)} \int_0^h \|u_{n(s)} + sh^{-1}d_{n(s)+1}\|_{L^4}^4 ds \\
&\leq h \sum_{n=0}^{m(t)} \left[\|u_n\|_{L^4}^4 + 2\|u_n\|_{L^4}^3 \|d_{n+1}\|_{L^4} + 2\|u_n\|_{L^4}^2 \|d_{n+1}\|_{L^4}^2 \right. \\
&\quad \left. + \|u_n\|_{L^4}^3 \|d_{n+1}\|_{L^4}^3 + \frac{1}{5}\|d_{n+1}\|_{L^4}^4 \right]
\end{aligned}$$

Theorem 13. *Let u be a mild solution of (3) with initial condition, φ the numerical approximation from Section 2.1, and Res the numerical approximation from (5). For $t \in [mh, (m+1)h]$ and $m \in \{1, \dots, M\}$ with $M = T/h$ we have for the error $r = u - \varphi + \text{Res}$ that*

$$\mathbb{E} [\|r(t)\|_{L^2}^2 | (Z_k)_{k \in \mathbb{N}}] \leq \|Q_N u_\star\|^2 + 2(\mathcal{K}_m)^4 + 6(\mathcal{K}_m)^2 (\mathcal{I}_m)^{1/2}$$

where the bound on the residual \mathcal{K}_m was defined in Theorem 10 and

$$\begin{aligned} \mathcal{I}_m = h \sum_{n=0}^m & \left[\|u_n\|_{L^4}^4 + 2\|u_n\|_{L^4}^3 \|d_{n+1}\|_{L^4} + 2\|u_n\|_{L^4}^2 \|d_{n+1}\|_{L^4} \right. \\ & \left. + \|u_n\|_{L^4}^3 \|d_{n+1}\|_{L^4}^3 + \frac{1}{5} \|d_{n+1}\|_{L^4}^4 \right] \end{aligned}$$

As we expect \mathcal{K}_m to be small, and the solution of the numerical scheme not, the second term should dominate in the error estimate. This is also confirmed in the numerical example.

Note that we usually neglect the error term coming from the initial condition by assuming that $Q_N u_\star = 0$. Anyway this can be made as small as we wish, by assuming that the initial condition is sufficiently smooth.

Let us also remark that r is not the error $d = u - \varphi$ we are interested in, but as we expect Res to be small, we neglect this in the following discussion.

6 Numerical Experiments

For the numerical result we focus on space-time white noise of strength 1, which means that all $\alpha_k = 1$. Moreover, as both the linear part and the nonlinearity are stable, we expect the solution to be of order 1 with rare events, where the solution is larger. Nevertheless, we expect solutions to be quite rough.

Due to poor regularity properties, we do not expect the numerical approximation to be very accurate, but still we tried a relatively poor discretization with $N = 128$ Fourier-modes and relatively large time-steps $h = 10^{-4}$ with a terminal time $T = 1$. Here we intend to show the robustness of the a-posteriori error estimate, which even for this poor discretization yields an almost reasonable result.

To simplify a little bit, we consider the initial condition $u_0 = u_\star = \sin(x)$ so that the projection to the high modes $Q_N u_\star = 0$, and we can neglect all error terms arising from the initial condition.

First in Figure 1 we plotted the residual \mathcal{K}_m^4 for $m = 1, \dots, 1/h$ together with the final error. As expected \mathcal{K}_m^4 is small and the error term from Theorem 13 is bounded by the error term involving \mathcal{K}_m^2 and the numerical data.

In Figure 2 we plot two terms of the residual \mathcal{K}_m^4 , for $m = 1, \dots, 1/h$. One of the main terms in \mathcal{K}_m^4 which depends on the numerical data is $\text{Res}_n^{\text{dat}}$,

therefore we plot $h \sum_{n=1}^m \|\text{Res}_n^{\text{dat}}\|_{L_4}^4$ in Figure 2(b) to see impact of this terms on the residual-bound \mathcal{K}_m^4 , which seems to be negligible.

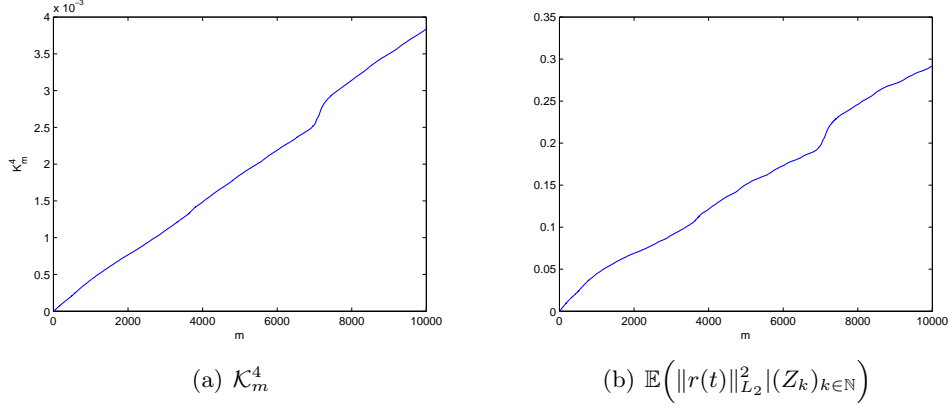


Figure 1: Comparison of the bound \mathcal{K}_m^4 on the residual and the final error bound from Theorem 13. Obviously, the \mathcal{K}_m^4 is not relevant in that estimate.

Moreover in Figure 2(a) we plot $h \sum_{n=0}^m \mathcal{S}_h(Z_{n+1} - e^{hA} Z_n)$, i.e, the term in \mathcal{K}_m^4 which arises from the OU-bridge. By comparing 1(a) and 2(a) we can see the impact of the OU-bridge on \mathcal{K}_m^4 . This gives a substantial, but not the most dominant term in \mathcal{K}_m^4 . We can also see that this error term is almost growing linear. The reason for this is that the part in \mathcal{S}_h that depends on the numerical data Z_n is quite small and the deterministic part of the estimate bounding the fluctuations dominates the bound.

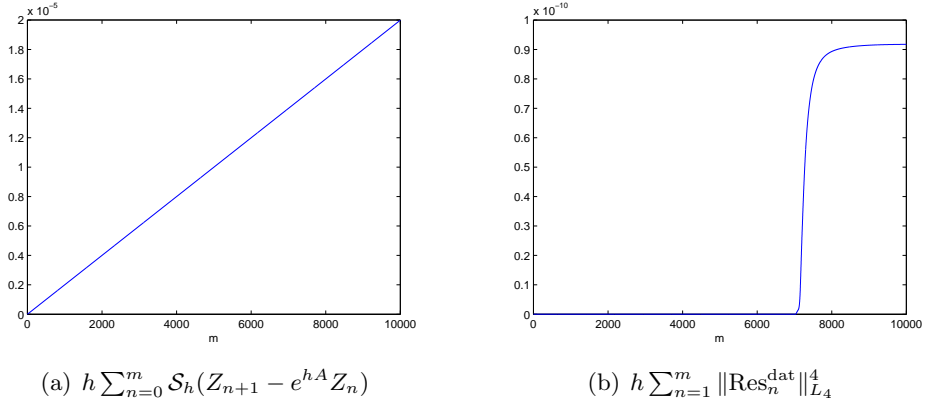


Figure 2: Values of $h \sum_{n=0}^m \mathcal{S}_h(Z_{n+1} - e^{hA} Z_n)$ which bounds the OU-bridge. This gives a substantial, but not the most dominant term in \mathcal{K}_m^4 . The data dependent terms $h \sum_{n=1}^m \|\text{Res}_n^{\text{dat}}\|_{L_4}^4$ at the discrete time-points in the residual are negligible. Occasionally these terms become suddenly larger, at points where we have a stronger increase in the error.

The final bound for the error $\mathbb{E} [\|r(mh)\|_{L_2}^2 | (Z_k)_{k \in \mathbb{N}}]$ which is stated in Theorem 13 is plotted in Figure 3 for 10 simulations. It confirms that (as expected) the numerical approximation we studied is poor and still far away from the asymptotic regime, but nevertheless it gives an almost meaningful result, as the error is smaller than the solution.

We also see in Figure 3 that the error is not growing with constant speed, but it has parts where it grows much faster. This effect is also very well visible in Figure 2(b), although the effect there is too small to have an impact on \mathcal{K}_m . We have no explanation for this behavior, which might be a large deviation effect, that actually might not be that rare due to noise strength of order one.

To see exactly impact of each term of \mathcal{K}_m in its value in Figure 4 values of \mathcal{K}_m , $m = 1, \dots, M$ and each of its terms are plotted. Also in Table 1 values of each term at the final time $T = 1$ is stated for 4 simulations.

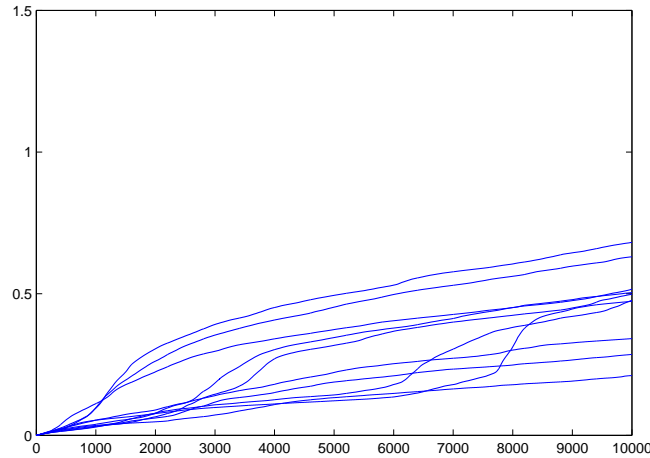


Figure 3: 10 simulations of our bound for $\mathbb{E}(\|r(t)\|_{L_2}^2 | (Z_k)_{k \in \mathbb{N}})$

References

- [1] A. Alabert, I. Gyöngy, *On numerical approximation of stochastic Burgers equation*, From stochastic calculus to mathematical finance, Springer, Berlin, 2006, 1–15.
- [2] S. Bartels, *A posteriori error analysis for time-dependent Ginzburg-Landau type equations*, Numer. Math. 99 (2005), no. 4, 557–583.

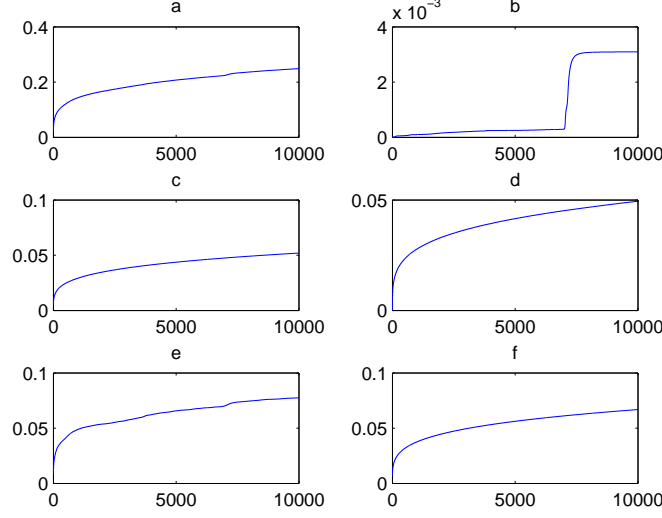


Figure 4: Values of (a). \mathcal{K}_m , (b). $\left(h \sum_{n=1}^m \|\text{Res}_n^{\text{dat}}\|_{L^4}^4\right)^{1/4}$, (c). $\left(\frac{3mh}{8\pi}\right)^{1/4} \left(\sum_{k>N} \frac{\alpha_k^2}{k^2}\right)^{1/2}$, (d). $\left(\frac{h^5}{5} \sum_{n=0}^m \|Au_n\|_{L^4}^4\right)^{1/4}$, (e). $\left(h^2 \sum_{n=0}^m \left[\frac{1}{2} \|u_n\|_{L^{12}}^3 + \dots + \frac{1}{20} \|d_{n+1}\|_{L^{12}}^3\right]\right)^{1/4}$, (f). $\left(h \sum_{n=0}^m \mathcal{S}_h(Z_{n+1} - e^{hA} Z_n)\right)^{1/4}$

- [3] S. Bartels, R. Müller, O. Christoph, *Robust a priori and a posteriori error analysis for the approximation of Allen-Cahn and Ginzburg-Landau equations past topological changes*, SIAM J. Numer. Anal. 49 (2011), no. 1, 110-134.
- [4] S. Becker, A. Jentzen, *Strong convergence rates for nonlinearity-truncated Euler-type approximations of stochastic Ginzburg-Landau equations*. ArXiv (2016).
- [5] D. Blömker, A. Jentzen, *Galerkin Approximations for the Stochastic Burgers Equation*, SIAM J. Numer. Anal. 51-1 (2013), 694-715.
- [6] D. Blömker, C. Nolde, J.C. Robinson, *Rigorous Numerical Verification of Uniqueness and Smoothness in a Surface Growth Model*, Journal of Mathematical Analysis and Applications 429(1):311–325, 2015.
- [7] D. Blömker, M. Romito, *Stochastic PDEs and lack of regularity (A surface growth equation with noise: existence, uniqueness, and blow-up)*

\mathcal{K}_m	0.2510	0.2611	0.2589	0.2545
E_1	0.0035	0.0059	0.0095	0.0023
E_2	0.05200	0.0520	0.0520	0.0519
E_3	0.0494	0.0494	0.0495	0.0495
E_4	0.0792	0.0870	0.0811	0.0838
E_5	0.06689	0.06691	0.066865	0.066865

Table 1: Values of four simulations of \mathcal{K}_m and each of it's term at the final time $T = 1$, i.e, $m = 10^4$. The contribution of error at discretization points is $E_1 = \left(h \sum_{n=1}^m \|\text{Res}_n^{\text{dat}}\|_{L^4}^4\right)^{1/4}$ for the part determined by the data and $E_2 = \left(\frac{3mh}{8\pi}\right)^{1/4} \left(\sum_{k>N} \frac{\alpha_k^2}{k^2}\right)^{1/2}$ for the stochastics on the high modes. The next two error terms $E_3 = \left(\frac{h^5}{5} \sum_{n=0}^m \|Au_n\|_{L^4}^4 ds\right)^{1/4}$ and $E_4 = \left(h^2 \sum_{n=0}^m \left[\frac{1}{2}\|u_n\|_{L^{12}}^3 + \dots + \frac{1}{20}\|d_{n+1}\|_{L^{12}}^3\right]\right)^{1/4}$ are data controlled terms for error that arises between discretization points. As the step size is quite large these terms seem to dominate although in h they seem to be of smaller order. Finally $E_5 = \left(h \sum_{n=0}^m \mathcal{S}_h(Z_{n+1} - e^{hA} Z_n)\right)^{1/4}$ bounding the stochastic fluctuation in between is not that large as we would expected it to be.

Jahresbericht der Deutschen Mathematiker-Vereinigung, 117(4):233-286, 2015.

- [8] T. Butler, C. Dawson, T. Wildey, *A posteriori error analysis of stochastic differential equations using polynomial chaos expansions*, SIAM J. Sci. Comput. 33 (2011), no. 3, 1267–1291.
- [9] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions. 2nd ed.*, vol. 152 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, (2014).
- [10] E.H. Georgoulis, C. Makridakis, *On a posteriori error control for the Allen-Cahn problem*, Math. Methods Appl. Sci. 37 (2014), no. 2, 173–179.
- [11] B. Goldys, B. Maslowski, *The OrnsteinUhlenbeck bridge and applications to Markov semigroups* Stoch. Proc. Appl. (2008) 18(10), 1738–1767.
- [12] A. Jentzen, P. Kloeden, G. Winkel, *Efficient simulation of nonlinear parabolic Spdes with additive noise*, Annals of Applied Probability. 21(3) (2011), 908–950.

- [13] E.A. Kalpinelli, N.E. Frangos, A.N. Yannacopoulos, *Numerical methods for hyperbolic SPDEs: a Wiener chaos approach*, Stoch. Partial Differ. Equ. Anal. Comput. 1 (2013), no. 4, 606–633.
- [14] P.E. Kloeden, G.J. Lord, A. Neuenkirch, T. Shardlow, *The exponential integrator scheme for stochastic partial differential equations: Pathwise error bounds*. J. Comput. Appl. Math. 235, No. 5, 1245.–1260 (2011).
- [15] G.J. Lord, C.E. Powell, T. Shardlow, *An introduction to computational stochastic PDEs*. Cambridge Texts in Applied Mathematics. Cambridge: Cambridge University Press, (2014).
- [16] C. Nolde, *Global regularity and uniqueness of solutions in a surface growth model using rigorous a-posteriori methods*. PhD-thesis, Universität Augsburg, 2017.
- [17] K.-Y. Moon, E. von Schwerin, A. Szepessy, R. Tempone, *An adaptive algorithm for ordinary, stochastic and partial differential equations*. Recent advances in adaptive computation, 325–343, Contemp. Math., 383, Amer. Math. Soc., Providence, RI, 2005
- [18] R. Verfürth, *A posteriori error estimation techniques for finite element methods*, in “Numerical Mathematics and Scientific Computation.” Oxford University Press, Oxford, 2013.
- [19] X. Yang, Y. Duan, Y. Guo, *A posteriori error estimates for finite element approximation of unsteady incompressible stochastic Navier-Stokes equations*, SIAM J. Numer. Anal. 48 (2010), no. 4, 15791600.
- [20] X. Yang, R. Qi, Y. Duan, *A posteriori analysis of finite element discretizations of stochastic partial differential delay equations*. J. Difference Equ. Appl. 18 (2012), no. 10, 16491663.